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Fuzzy line bundles, the Chern character and topological charges over the fuzzy sphere

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Abstract

Using the theory of quantized equivariant vector bundles over compact coadjoint orbits we determine the Chern characters of all noncommutative line bundles over the fuzzy sphere with regard to its derivation-based differential calculus. The associated Chern numbers (topological charges) arise to be noninteger, in the commutative limit the well-known integer Chern numbers of the complex line bundles over the two-sphere are recovered.

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1. Introduction and overview

Classical gauge field theories exhibit interesting features connected with the geometry and topology of nontrivial fiber bundles (over space or space–time). Examples are monopole and instanton solutions.

The Serre–Swan theorem [1] (cf. [2,3]) leads to a complete equivalence between the category of continuous vector bundles over a compact manifold M and the finitely generated projective modules over the unital commutative C^*-algebra $C(M)$ of continuous functions over M. This circumstance can be generalized to the smooth case [3] and even holds for noncompact manifolds. Moreover, the geometry of M is encoded in $C(M)$.

In noncommutative geometry one proceeds by *defining* vector bundles as finitely generated projective left, right or bimodules over some algebra, which is thought of as the

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algebra of functions over some noncommutative manifold. Accordingly, the *geometrical nontriviality* is purely algebraical and encoded solely in projective nonfree modules over the noncommutative algebra under consideration.

For the fuzzy sphere this case has first been analyzed in [4](see also [5]), leading to scalar and spinor field configurations in monopole backgrounds. A different approach using spectral triples and their Dirac operator-based differential calculi has been used in [6,7]. This leads to integer topological charges similar to [4].

We review the definition of the Chern character on projective modules in Section 2 and the complex line bundles over the two-sphere in Section 3. In Sections 4 and 5, we will show that the Chern character of projective modules over the matrix algebra of the fuzzy sphere gives, with respect to its free derivation-based differential calculus, rise to noninteger Chern numbers. In the commutative limit the well-known Chern characters of complex line bundles over the two-sphere with its integer topological charges are recovered.

2. The Chern character of projective modules

Let $\mathcal{A}$ be a complex unital not necessarily commutative C^*-algebra and denote $\mathcal{A}\otimes\mathbb{C}^n$ by $\mathcal{A}^n$. Then any projector (self-adjoint idempotent) $p \in M_n(\mathcal{A})$, where $M_n(\mathcal{A}) = A \otimes M_n(\mathbb{C})$ denotes the $n \times n$-matrices with coefficients in $\mathcal{A}$, defines a (finitely generated) projective left $\mathcal{A}$-module by $E = \mathcal{A}^n p$. Elements ψ of E can be viewed as $\psi \in \mathcal{A}^n$ with $\psi p = \psi$. If $\mathcal{A}$ is further endowed with a differential calculus $(\Omega^*(\mathcal{A}), \mathrm{d})$, the Grassmann connection $\nabla : E \to \Omega^1(\mathcal{A})\otimes_{\mathcal{A}} E$ of E is defined by $\nabla = p\circ \mathrm{d}$. It satisfies $\nabla(f\psi) = f\,\nabla\psi + \mathrm{d}f\otimes_{\mathcal{A}}\psi$ for all $f \in \mathcal{A}$, $\psi \in E$. After extending ∇ to $\Omega^1(\mathcal{A}) \otimes_{\mathcal{A}} E$ one can define the $\mathcal{A}$-linear map

$$\nabla^2 : E \to \Omega^2(\mathcal{A}) \otimes_{\mathcal{A}} E, \tag{1}$$

called the curvature of ∇. For more details, see [8,9]. Evaluating ∇^2 one finds $\nabla^2 = p(\mathrm{d}p)(\mathrm{d}p)$, whereas if we write $(p_{ij}) = p \in M_n(\mathcal{A})$, $(\mathrm{d}p)$ is the $n \times n$-matrix with coefficients $\mathrm{d}p_{ij}$ and the entries of $p(\mathrm{d}p)(\mathrm{d}p)$ are $p_{il}\,\mathrm{d}p_{lk} \wedge \mathrm{d}p_{kj}$. The curvature can be viewed as $\nabla^2 \in \Omega^2(\mathcal{A}) \otimes_{\mathcal{A}} \mathrm{End}_{\mathcal{A}}(E)$, where $\mathrm{End}_{\mathcal{A}}(E)$ denotes the right $\mathcal{A}$-module of endomorphisms of E, i.e. $\mathcal{A}$-linear mappings from E to E. Now define

$$\mathbf{F} := \mathrm{Tr}\, p(\mathrm{d}p)(\mathrm{d}p) \in \Omega^2(\mathcal{A}), \tag{2}$$

which is a cocycle, i.e. $\mathrm{d}\mathbf{F} = \mathrm{Tr}(\mathrm{d}p)(\mathrm{d}p)(\mathrm{d}p) = 0$. Here Tr is the trace in $\mathrm{End}_{\mathcal{A}}$. So $\mathbf{F}$ defines a cohomology class $[\mathbf{F}] \in H^2(\mathcal{A})$. More generally, the Chern character of E with respect to $(\Omega^*(\mathcal{A}), \mathrm{d})$ is the set of

$$\mathrm{Ch}_r(p) := \frac{1}{r!}\mathrm{Tr}\, p(\mathrm{d}p)^{2r}, \quad r \in \mathbb{N} \cup \{0\},$$

where $\mathrm{Ch}_r(p)$ are called its rth components. They are also cocycles and provide equivalence classes in $H^{2r}(\mathcal{A})$. $\mathrm{Ch}_0(p) = \mathrm{Tr}\, p$ simply gives the rank of the module.

3. Complex line bundles over the two-sphere

One approach to the construction of the complex line bundles over the two-sphere is the one given in [10], cf. also [11]. Starting with the complex Hopf fiberation $\mathrm{U}(1) \hookrightarrow \mathrm{SU}(2) \simeq S^3 \twoheadrightarrow S^2$ and the irreducible representations of U(1) on $\mathbb{C}$ labeled by integers $k \in \mathbb{Z}$, one defines the space of smooth equivariant functions $C^\infty_{(k)}(S^3, \mathbb{C}) \ni \varphi : S^3 \to \mathbb{C}$ with $\varphi(x \cdot z) = z^{-k}\varphi(x)$ for $x \in S^3$ and $z \in \mathrm{U}(1)$. These are modules over $C^\infty_{(0)}(S^3, \mathbb{C}) \simeq C^\infty(S^2, \mathbb{C})$, and as such are isomorphic the smooth sections $\Gamma^\infty(S^2, L^k)$ of the associated complex line bundles $L^k := S^3 \times_k \mathbb{C}$ over the two-sphere.

By the Serre–Swan theorem these modules are finitely generated and projective and hence, it is possible to identify $\Gamma^\infty(S^2, L^k)$ with $(C^\infty(S^2, \mathbb{C}))^n p$, where $p \in M_n(C^\infty(S^2, \mathbb{C}))$ is a projector. In [10] the projectors p where explicitly constructed with the help of the equivariant functions $C^\infty_{(k)}(S^3, \mathbb{C})$. The integer $n \in \mathbb{N}$ turned out to be $|k| + 1$ and the first Chern numbers where calculated giving

$$c_1(p) := -\frac{1}{2\pi \mathrm{i}} \int_{S^2} \mathrm{Tr}\, p(\mathrm{d}p)(\mathrm{d}p) = -k \in \mathbb{Z}.$$

Let us shortly mention that k is related to the magnetic charge Q_m of a Dirac (point) monopole in $\mathbb{R}^3$ via

$$Q_\mathrm{m} = k\frac{\hbar c}{2e}, \tag{3}$$

where $\hbar$ is the Planck's constant over 2π, c the vacuum speed of light and e is the elementary electrical charge, meaning that Q_m is quantized.

4. Fuzzy line bundles

4.1. General remarks

We start with the repetition of well-known facts about the fuzzy sphere and its free derivation-based differential calculus. Then we use the prescription of quantizing equivariant vector bundles over coadjoint orbits to obtain the projectors that define the modules over the matrix algebra of the fuzzy sphere and its Chern characters.

Denote SU(2) by G, its Lie algebra su(2) by $\mathfrak{g}$ and let $\{X_a\}_{a=1,2,3}$ be the generators of the irreducible spin-N representation of $\mathfrak{g}$ acting on the representation space $[N]$ with dim $([N]) = 2N + 1$.

The algebra of the fuzzy sphere [12,13] is the noncommutative algebra $\mathrm{End}([N]) =: \mathcal{A}_N$, the algebra of $(2N + 1) \times (2N + 1)$-matrices with complex coefficients. $\mathcal{A}_N$ is generated by $Y_a = (N(N + 1))^{-1/2} X_a$ which satisfy

$$[Y_a, Y_b] = \frac{\mathrm{i}\epsilon_{abc}}{\sqrt{N(N+1)}} Y_c \quad \text{and} \quad \sum_{a=1}^{3} (Y_a)^2 = 1. \tag{4}$$

The derivation-based differential calculus (cf. [14]) on $\mathcal{A}_N$ is defined as follows. One chooses the three derivations ("vector fields") e_a, defined by $e_a(\phi) := [X_a, \phi]$ for $\phi \in \mathcal{A}_N$. We denote by $\mathrm{Der}_3(\mathcal{A}_N)$ the linear subspace of $\mathrm{Der}(\mathcal{A}_N)$ generated by the e_a's. Here $\mathrm{Der}(\mathcal{A}_N)$ is the $\mathbb{C}$-vector space of all derivations of $\mathcal{A}_N$. $\mathcal{A}_N$ decomposes into $[0] \oplus [1] \oplus \cdots \oplus [2N]$ as $\mathfrak{g}$- and G-module, respectively. The derivations e_a satisfy $[e_a, e_b] = \mathrm{i}\epsilon_{abc} e_c$, and are the noncommutative analogue of the vector fields $L_a = \mathrm{i}\epsilon_{abc} y_b \partial/\partial y_c$ on the two-sphere. The set of p-forms $\Omega^p_{(N)}$ over $\mathcal{A}_N$ is the free $\mathcal{A}_N$-module:

$$\Omega^p_{(N)} = \mathcal{A}_N \otimes (\mathrm{Der}_3(\mathcal{A}_N)^* \wedge \cdots \wedge \mathrm{Der}_3(\mathcal{A}_N)^*) \simeq \mathcal{A}_N \otimes (\mathfrak{g}^*_{\mathbb{C}} \wedge \cdots \wedge \mathfrak{g}^*_{\mathbb{C}}),$$

where $\mathfrak{g}_{\mathbb{C}} \simeq \mathrm{sl}(2, \mathbb{C})$ is the complexification of $\mathfrak{g}$. Note that $\Omega^p_{(N)} = 0$ for $p > 3$. The exterior derivative $\mathrm{d} : \mathcal{A}_N \to \Omega^1_{(N)}$ is defined by $\mathrm{d}\phi(u) = u(\phi)$ for all $\phi \in \mathcal{A}_N$ and $u \in \mathrm{Der}_3(\mathcal{A}_N)$. It extends to $\Omega^*_{(N)} = \oplus_p \Omega^p_{(N)}$ by linearity and the graded Leibniz rule. There is a distinguished one-form Θ defined by $\Theta(e_a) = -X_a$. Θ is the analogue of the Maurer–Cartan form satisfying $\mathrm{d}\Theta + \Theta^2 = 0$. The exterior derivative of a zero form $\phi \in \Omega^0_{(N)} = \mathcal{A}_N$ can with the help of Θ be written as $\mathrm{d}\phi = -[\Theta, \phi]$. One can choose a basis Θ_a in $\Omega^1_{(N)}$ completely determined by $\Theta_a(e_b) = \delta_{ab}\mathbb{I}$. Then $\Theta = -X_a \Theta_a$ and $\mathrm{d}\phi = e_a(\phi)\Theta_a$. It follows from the procedure given in [15] (cf. also [16]) that the quantization of equivariant vector bundles over coadjoint orbits is achieved by means of the orthogonal projection

$$p \in [N] \otimes [N]^* \otimes [\nu] \otimes [\nu]^* \cong \mathcal{A}_N \otimes \mathrm{End}[\nu], \tag{5}$$

which projects onto the unique irreducible subrepresentation of $[N] \otimes [\nu]$ with highest (lowest) spin, i.e. onto $[N \pm \nu]$. Here $[\nu]$ is the representation space of the irreducible spin-ν representation of $\mathfrak{g}$. In the case $[N - \nu]$ we have to assume that $N > \nu$. So the fuzzy line bundles obtained in this way are of the form $\mathbf{L}^{\pm 2\nu} := (\mathcal{A}_N \otimes [\nu])p$, they are isomorphic to the $(2N+1) \times (2(N \pm \nu)+1)$-matrices. Here p acts from the right providing left $\mathcal{A}_N$-modules. The modules $\mathbf{L}^k$ approximate in the commutative limit $N \to \infty$ the modules of sections of L^k.

Let $\pi : G \to \mathrm{End}([N + \nu])$ be the irreducible representation of G with spin $N \pm \nu$ and $|h\rangle$ its highest weight vector. $|h\rangle$ is thought of as being embedded in $[N] \otimes [\nu]$ by $|h\rangle \oplus 0 \oplus \cdots \oplus 0$. Denote by μ the normalized Haar measure on G.

Lemma 1. *The projector* $p : [N] \times [\nu] \to [N \pm \nu]$ *defined above is given by*

$$p = (2(N \pm \nu) + 1) \int_G \pi(g)|h\rangle\langle h|\pi(g)^{-1}\, \mathrm{d}\mu(g). \tag{6}$$

Proof. Denote by p_1 the right-hand side of Eq. (6). Then p_1 sends every vector in $[N \pm \nu]^\perp$ to zero. Now the invariance of μ implies that $\pi(g)p_1\pi(g)^{-1} = p_1\ \forall g \in G$, so by the Schur lemma p_1 is proportional to the identity on $[N \pm \nu]$. Since $\mathrm{Tr}\, p_1 = 2(N \pm \nu) + 1$, p_1 restricted to $[N \pm \nu]$ is the identity and $p_1^2 = p_1$. Accordingly, $p = p_1$. □

4.2. Explicit calculations

For the sake of simplicity we identify in this section $\mathrm{Der}_3(\mathcal{A}_N)$ with $\mathfrak{g}_{\mathbb{C}}$. It is now our aim to calculate the first component of the Chern character determined by p, i.e. $\mathbf{F} = \mathrm{Tr}_2(p\,\mathrm{d}p\,\mathrm{d}p) \in \mathcal{A}_N \otimes (\mathfrak{g}_{\mathbb{C}}^* \wedge \mathfrak{g}_{\mathbb{C}}^*)$, where 'd' acts only on the $\mathcal{A}_N$ part of $p \in \mathcal{A}_N \otimes \mathrm{End}([\nu])$ and Tr_2 is the trace in $\mathrm{End}([\nu])$.

Lemma 2. $\mathbf{F} = f\epsilon_{abc}X_c\Theta_a \wedge \Theta_b$ *with* $f \in \mathbb{C}\mathbb{I}$.

Proof. Let Ad be the adjoint representation of G on $\mathfrak{g}_{\mathbb{C}} \ni u \mapsto \mathrm{Ad}_g u = gug^{-1}$ and Θ as defined in Section 4.1. Then $\Theta \otimes \mathbb{I} \in \Omega^1_{(N)} \otimes \mathrm{End}([\nu])$ transforms as

$$\Theta(\mathrm{Ad}_g u) \otimes \mathbb{I} = (\pi_1(g) \otimes \mathbb{I})(\Theta(u) \otimes \mathbb{I})(\pi_1^{-1}(g) \otimes \mathbb{I}) = \bar{\pi}(g)(\Theta(u) \otimes \mathbb{I})\bar{\pi}^{-1}(g),$$

where $\bar{\pi} = \pi_1 \otimes \pi_2$ is acting on $[N] \otimes [\nu]$ and $u \in \mathfrak{g}_{\mathbb{C}}$, i.e. $\Theta \otimes \mathbb{I}$ is invariant under the action of G on $\Omega^1_{(N)} \otimes \mathrm{End}([\nu])$. This implies for $\mathrm{d}p = -[\Theta \otimes \mathbb{I}, p]$ that $\mathrm{d}p(\mathrm{Ad}_g u) = \bar{\pi}(g)\,\mathrm{d}p(u)\,\bar{\pi}^{-1}(g)$ and finally for the first component of the Chern character

$$\mathbf{F}(u, v) = \pi_1(g)\mathbf{F}(\mathrm{Ad}_{g^{-1}}u, \mathrm{Ad}_{g^{-1}}v)\pi_1^{-1}(g)$$

for all $u, v \in \mathfrak{g}_{\mathbb{C}}$. So $\mathbf{F}$ is an invariant element of $\mathcal{A}_N \otimes (\mathfrak{g}_{\mathbb{C}}^* \wedge \mathfrak{g}_{\mathbb{C}}^*)$. Reducing this latter space as a G-module shows that [0] appears only once, as $\mathfrak{g}_{\mathbb{C}}^* \wedge \mathfrak{g}_{\mathbb{C}}^* \simeq [1]$. Consequently, the subspace of invariant two-forms is one dimensional, and since $\epsilon_{abc}X_c\Theta_a \wedge \Theta_b$ is invariant, $\mathbf{F}$ can be written as claimed. □

Note that $f\epsilon_{abc}X_c\Theta_a \wedge \Theta_b$ can also be written as $\mathrm{i}q/4\epsilon_{abc}Y_a\,\mathrm{d}Y_b \wedge \mathrm{d}Y_c$, where q and f are related by

$$q = \frac{4}{\mathrm{i}}\frac{(N(N+1))^{3/2}}{1/2 - N(N+1)}f,$$

as can be seen by expanding $\mathrm{d}Y_a = [X_b, Y_a]\Theta_b$. It will turn out later that q can be naturally interpreted as Chern numbers.

What is left to do is to determine f, depending on N and ν. For this note first that

$$\mathrm{Tr}_2(p\,\mathrm{d}p(e_a)\,\mathrm{d}p(e_b)) = f\epsilon_{abd}X_d.$$

Now multiply this equation with $\epsilon_{abc}X_c$ and take the trace also in $\mathcal{A}_N$ to get

$$f = \frac{\epsilon_{abc}\mathrm{Tr}(p\,\mathrm{d}p(e_a)\,\mathrm{d}p(e_b)X_c)}{2N(N+1)(2N+1)}, \tag{7}$$

where Tr denotes the trace in $\mathrm{End}([N]) \otimes \mathrm{End}([\nu])$. This expression can be further simplified by the following lemma.

Lemma 3. $\epsilon_{abc}\,\mathrm{Tr}(p\,\mathrm{d}p(e_a)\,\mathrm{d}p(e_b)X_c) = (2(N \pm \nu) + 1)\epsilon_{abc}\langle h|[X_a, p][X_b, p]X_c|h\rangle$.

Proof. With $\bar{\pi}$ as above, the left-hand side is by Lemma 1:

$$(2(N \pm \nu) + 1)\int_G \epsilon_{abc}\langle h|\bar{\pi}^{-1}(g)[X_a, p][X_b, p]X_c\bar{\pi}(g)|h\rangle\,\mathrm{d}\mu(g). \tag{8}$$

Now $\epsilon_{abc}[X_a, p][X_b, p]X_c$ are the components of the equivariant multilinear map $-[\Theta \otimes \mathbb{I}, p][\Theta \otimes \mathbb{I}, p](\Theta \otimes \mathbb{I})$ from $\mathfrak{g}_{\mathbb{C}} \wedge \mathfrak{g}_{\mathbb{C}} \wedge \mathfrak{g}_{\mathbb{C}} \simeq [0]$ to $\mathcal{A}_N \otimes \mathrm{End}[\nu])$ which is constant. This implies the assertion. □

Let us denote the expectation value $\epsilon_{abc}\langle h| \cdot |h\rangle$ appearing in Lemma 3 by B. Expanding the commutators it is straightforward to see that $B = C + \mathrm{i}D$ with

$$C = \epsilon_{abc}\langle h|X_a p X_b p X_c|h\rangle \quad \text{and} \quad D = -\langle h|X_a p X_a|h\rangle. \tag{9}$$

Consider now (cf. [17]) the space of homogenous polynomials $\mathcal{H}_n$ of two complex variables z_1 and z_2 of fixed degree $n \in \mathbb{N}$. We define the following "creation" and "annihilation" operators $a_i^\dagger = z_i$ and $a_i = \partial/\partial z_i$, satisfying $[a_i, a_j^\dagger] = \delta_{ij}$, which give an irreducible representation of $\mathfrak{g}$ by

$$X_1 = \tfrac{1}{2}(a_1^\dagger a_2 + a_2^\dagger a_1), \quad X_2 = -\tfrac{1}{2}\mathrm{i}(a_1^\dagger a_2 + a_2^\dagger a_1) \quad \text{and} \quad X_3 = \tfrac{1}{2}(a_1^\dagger a_2 + a_2^\dagger a_2)$$

with spin $\frac{1}{2}n$ on $\mathcal{H}_n$. To compute C and D of Eq. (9) we realize the $[N] \otimes [\nu]$ representation on $\mathcal{H}_n \otimes \mathcal{H}_l$ with $2N = n$ and $2\nu = l$, respectively. An orthonormal basis of $\mathcal{H}_n$ is given by

$$|\psi_k\rangle = \binom{n}{k}^{1/2} z_1^k z_2^{n-k} \quad \text{with} \quad \langle \psi_k|\psi_{k'}\rangle = \delta_{kk'}. \tag{10}$$

First we analyze the case where p projects onto $[N + \nu]$. Then the highest weight vector $|h\rangle \in \mathcal{H}_n \otimes \mathcal{H}_l$ is given by $|h\rangle = z_1^n \otimes z_1^l$, i.e. $(X_3 \otimes \mathbb{I} + \mathbb{I} \otimes X_3)|h\rangle = \frac{1}{2}(n + l)|h\rangle$ and $(X_+ \otimes \mathbb{I} + \mathbb{I} \otimes X_+)|h\rangle = 0$, with $\|h\| = 1$. Define $|w\rangle := X_1|h\rangle$, then $X_2|h\rangle = \mathrm{i}|w\rangle$. Since $X_3|h\rangle = \frac{1}{2}n|h\rangle$ a lengthy but straightforward calculation yields

$$B = 2\mathrm{i}(n-1)\langle w|p|w\rangle - 2\mathrm{i}\langle w|pX_3p|w\rangle - \tfrac{1}{4}\mathrm{i}n^2. \tag{11}$$

Now because $|w\rangle$ and $|v\rangle := (X_- \otimes \mathbb{I} + \mathbb{I} \otimes X_-)|h\rangle$ have the same eigenvalue $\frac{1}{2}(n + l - 2)$ of $X_3 \otimes \mathbb{I} + \mathbb{I} \otimes X_3$, we know that $p|w\rangle = \lambda|v\rangle$. Accordingly, λ can be evaluated

$$\lambda = \frac{\langle v|w\rangle}{\langle v|v\rangle} = \frac{1}{2}\frac{n}{n+l}.$$

This gives

$$\langle w|p|w\rangle = \frac{1}{4}\frac{n^2}{n+l} \quad \text{and} \quad \langle w|pX_3p|w\rangle = \frac{1}{8}\frac{n^2}{(n+l)^2}(n(n-2) + nl). \tag{12}$$

Inserting Eq. (12) into Eq. (11) this gives finally for f in Eq. (7) expressed in terms of N and ν

$$f = -\mathrm{i}N\nu\frac{(N+\nu+1)(N+\nu+1/2)}{(N+\nu)^2(2N+1)(N+1)}. \tag{13}$$

The case where p projects onto $[N - \nu]$ is more involved, since we first have to determine the highest weight vector. The ansatz

$$|h\rangle = \sum_{k=0}^{l} a_k z_2^k z_1^{n-k} \otimes z_1^k z_2^{l-k}$$

leads through $(X_+ \otimes \mathbb{I} + \mathbb{I} \otimes X_+)|h\rangle = 0$ to the recursion relation $(k-l)a_k = (k+1)a_{k+1}$ which is solved by

$$a_k = (-1)^k a_0 \binom{l}{k}. \tag{14}$$

The remaining a_0 is determined by the normalization condition $\|h\| = 1$ and gives for a_k

$$a_k = (-1)^k \sqrt{\frac{n-l+1}{n+1}} \binom{l}{k} \quad \text{for} \quad k = 0, \dots, l, \tag{15}$$

where we have used the formula

$$\sum_{k=0}^{l} \binom{l}{k} \binom{n}{k}^{-1} = \frac{n+1}{n-l+1}. \tag{16}$$

Proceeding analogously we define $|w\rangle = X_1|h\rangle$, but now $|w\rangle = |w_+\rangle + |w_-\rangle$ with

$$(X_3 \otimes \mathbb{I} + \mathbb{I} \otimes X_3)|w_\pm\rangle = (\tfrac{1}{2}(n-1) \pm 1)|w_\pm\rangle.$$

From this it follows that $pX_1|h\rangle = p|w_-\rangle = \lambda|v\rangle$ and $pX_2|h\rangle = \mathrm{i}p|w_-\rangle = \mathrm{i}\lambda|v\rangle$. Using

$$\sum_{k=0}^{l} \binom{l}{k} \binom{n}{k+1}^{-1} = \frac{n+1}{(n-l)(n+1-l)} \quad \text{and} \quad \sum_{k=0}^{l} (n-k) \binom{l}{k} \binom{n}{k+1}^{-1} = \frac{(n+1)(n+2)}{(l-n-1)(l-n-2)} \tag{17}$$

one finds with $(X_- \otimes \mathbb{I} + \mathbb{I} \otimes X_-)|h\rangle =: |v\rangle$ for the proportionality factor λ

$$\lambda = \frac{\langle v|w_-\rangle}{\langle v|v\rangle} = \frac{1}{2}\frac{n+2}{n-l+2}. \tag{18}$$

Applying X_3 to $|h\rangle$ yields $\frac{1}{2}n|h\rangle - |K\rangle$ with

$$|K\rangle = \sum_{k=1}^{l} a_k k z_2^k z_1^{n-k} \otimes z_1^k z_2^{l-k}.$$

Since $(X_3 \otimes \mathbb{I} + \mathbb{I} \otimes X_3)|K\rangle = \frac{1}{2}(n-l)|K\rangle$ it follows that $pX_3|h\rangle = (\frac{1}{2}n - \mu)|h\rangle$, where μ is given by

$$\mu = \langle h|K\rangle = \frac{l}{n+2-l},$$

for which a formula similar to Eq. (17) has been used. Now we have enough ingredients to write B as

$$B = 2\mathrm{i}\lambda^2((n-2\mu-1)\langle v|v\rangle - \langle v|X_3|v\rangle) - \mathrm{i}(\tfrac{1}{2}n-\mu)^2. \tag{19}$$

It is left to determine $\langle v|X_3|v\rangle$. One finds

$$\begin{aligned}\langle v|X_3|v\rangle &= \frac{n+1-l}{n+1}\frac{(n-l)^2}{2}\sum_{k=0}^{l}\binom{l}{k}\binom{n}{k+1}^{-1}(n-2k-2)\\ &= \frac{(n+2)(n-l-2)(n-l)}{2(n-l+2)}.\end{aligned}$$

We collect the results to obtain f in terms of N and v

$$f = \frac{\mathrm{i}\nu(N+1)(N-\nu)(2N-2\nu+1)}{2N(2N+1)(N-\nu+1)^2}. \tag{20}$$

5. Results and commutative limit

Summarizing the calculations of the previous section we get for the Chern character $\mathbf{F}$ of the modules $\mathbf{L}^k$, with $k=\pm 2\nu$, the formula

$$\mathbf{F} = \tfrac{1}{4}\mathrm{i}q\epsilon_{abc}Y_a\,\mathrm{d}Y_b\wedge\mathrm{d}Y_c, \tag{21}$$

where

$$q = \frac{4}{\mathrm{i}}\frac{(N(N+1))^{3/2}}{1/2-N(N+1)}f, \tag{22}$$

and

$$f = \frac{\mathrm{i}\nu(N+1)(N-\nu)(2N-2\nu+1)}{2N(2N+1)(N-\nu+1)^2} \quad \text{for} \quad k=2\nu>0, \tag{23}$$

$$f = -\mathrm{i}N\nu\frac{(N+\nu+1)(N+\nu+1/2)}{(N+\nu)^2(2N+1)(N+1)} \quad \text{for} \quad k=-2\nu>0. \tag{24}$$

What is needed to obtain the associated Chern numbers is a certain notion of integration over two-forms. Thus, for any $\phi\in\mathcal{A}_N$ and

$$\omega := \frac{\epsilon_{abc}Y_a}{8\pi}\,\mathrm{d}Y_b\wedge\mathrm{d}Y_c\in\Omega^2(\mathcal{A}_N),$$

we define the integral by

$$\int\phi\omega := \mathrm{Tr}_N(\phi),\qquad \mathrm{Tr}_N(\cdot) = \frac{1}{2N+1}\mathrm{Tr}(\cdot), \tag{25}$$

with $\int\omega=1$. The two-form ω is the noncommutative volume form, which in the commutative limit converges to the normalized volume form on S^2. Consequently, the first Chern numbers of the fuzzy line bundles determined by p are given by

$$c_1(p) := -\frac{1}{2\pi\mathrm{i}}\int\mathbf{F} = -q. \tag{26}$$

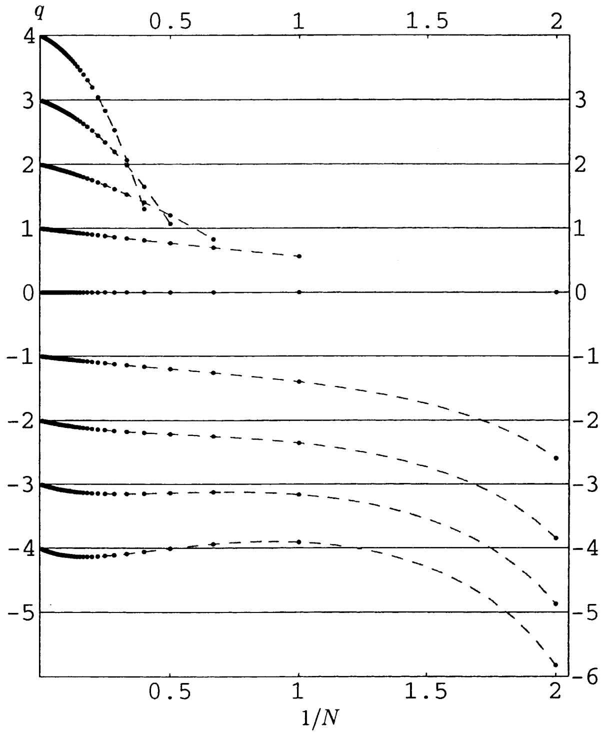

Fig. 1. Topological charges q with commutative limit k between -4 and 4 as function of the fuzzyness $1/N$. These can be viewed as the (fuzzy) magnetic charges of a Dirac monopole living on the fuzzy sphere. Dashed lines connect charges of constant ν.

In the commutative limit we find for the topological charges

$$k = \lim_{N\to\infty} c_1(p) = \mp 2\nu \in \mathbb{Z}, \tag{27}$$

where the minus sign corresponds to the projection onto $[N+\nu]$ and the plus sign to $[N-\nu]$. Some topological charges q and their commutative limits k are shown in Fig. 1.

6. Conclusions

We constructed projective modules over the matrix algebra $\mathcal{A}_N$ of the fuzzy sphere using the prescription of quantizing equivariant vector bundles given in [15], leading to fuzzy line bundles. With respect to the free derivation-based differential calculus $(\Omega^*_{(N)}, \mathrm{d})$ on $\mathcal{A}_N$ we calculated the Chern character $\mathbf{F} \in \Omega^2_{(N)}$. Since $\mathbf{F}$ was seen to be SU(2)-invariant, i.e. an SU(2)-equivariant mapping from $\mathrm{sl}(2,\mathbb{C}) \wedge \mathrm{sl}(2,\mathbb{C})$ to $\mathcal{A}_N$, it was unique up to a factor. The determination of this factor $f \in \mathbb{C}$ was achieved and with the help of a certain notion of integration the Chern numbers q associated with $\mathbf{F}$ were calculated. These turned out to be noninteger, becoming integers in the commutative limit $N \to \infty$.

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